NEW NORMAL SUBGROUPS FOR THE GROUP REPRESENTATION OF THE CAYLEY TREE.

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ABSTRACT. In this paper we give full description of normal subgroups of index eight and ten for the group.

Key words: G_k- group, normal subgroup, homomorphism, epimorphism.

1. INTRODUCTION

There are several thousand papers and books devoted to the theory of groups. But still there are unsolved problems, most of which arise in solving of problems of natural sciences as physics, biology etc. In particular, if configuration of physical system is located on a lattice (in our case on the graph of a group) then the configuration can be considered as a function defined on the lattice. There are many works devoted to several kind of partitions of groups (lattices) (see e.g. [1], [3], [5], [7]).

One of the central problems in the theory of Gibbs measures is to study periodic Gibbs measures corresponding to a given Hamiltonian. It is known that there exists a one to one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k(see [5]). For any normal subgroups H of the group G_k we define H-periodic Gibbs measures.

In Chapter 1 of [5] it was constructed several normal subgroups of the group representation of the Cayley tree. In [4] we found full description of normal subgroups of index four and six for the group. In this paper we continue this investigation and construct all normal subgroups of index eight and ten for the group representation of the Cayley tree.

Cayley tree. A Cayley tree (Bethe lattice) Γ^k of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k + 1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L that of edges (arcs).

A group representation of the Cayley tree. Let G_k be a free product of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, ...a_{k+1}$, respectively.

It is known that there exists a one to one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

To give this correspondence we fix an arbitrary element $x_0 \in V$ and let it correspond to the unit element e of the group G_k. Using $a_1, ..., a_{k+1}$ we numerate the nearest-neighbors

of element e, moving by positive direction. Now we'll give numeration of the nearest-neighbors of each $a_i, i = 1, ..., k+1$ by $a_i a_j, j = 1, ..., k+1$. Since all a_i have the common neighbor e we give to it $a_i a_i = a_i^2 = e$. Other neighbor are numerated starting from $a_i a_i$ by the positive direction. We numerate the set of all the nearest-neighbors of each $a_i a_j$ by words $a_i a_j a_q, q = 1, ..., k+1$, starting from $a_i a_j a_j = a_i$ by the positive direction. Iterating this argument one gets a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

Any(minimal represented) element $x \in G_k$ has the following form: $x = a_{i_1} a_{i_2}...a_{i_n}$, where $1 \leq i_m \leq k+1, m = 1, ..., n$. The number n is called the length of the word x and is denoted by $l(x)$. The number of letters $a_i, i = 1, ..., k+1$, that enter the non-contractible representation of the word x is denoted by $w_x(a_i)$.

Proposition 1. *[2] Let φ be homomorphism of the group G_k with the kernel H. Then H is a normal subgroup of the group G_k and $\varphi(G_k) \simeq G_k/H$, (where G_k/H is a quotient group) i.e., the index $|G_k : H|$ coincides with the order $|\varphi(G_k)|$ of the group $\varphi(G_k)$.*

Let H be a normal subgroup of a group G. Natural homomorphism g from G onto the quotient group G/H by the formula $g(a) = aH$ for all $a \in G$. Then $Ker\varphi = H$.

Definition 1. *Let $M_1, M_2, ..., M_m$ be some sets and $M_i \neq M_j$, for $i \neq j$. We call the intersection $\cap_{i=1}^m M_i$ contractible if there exists $i_0 (1 \leq i_0 \leq m)$ such that*

$$\cap_{i=1}^m M_i = \left(\cap_{i=1}^{i_0 - 1} M_i \right) \cap \left(\cap_{i=i_0+1}^m M_i \right).$$

Let $N_k = \{1, ..., k+1\}$. The following Proposition describes several normal subgroups of G_k.

Put

$$H_A = \left\{ x \in G_k \mid \sum_{i \in A} \omega_x(a_i) \text{ is even} \right\}, \quad A \subset N_k. \tag{1.1}$$

Proposition 2. *[5] For any $\emptyset \neq A \subseteq N_k$, the set $H_A \subset G_k$ satisfies the following properties:*
(a) H_A is a normal subgroup and $|G_k : H_A| = 2$;
(b) $H_A \neq H_B$, for all $A \neq B \subseteq N_k$;
(c) Let $A_1, A_2, ..., A_m \subseteq N_k$. If $\cap_{i=1}^m H_{A_i}$ is non-contractible, then it is a normal subgroup of index 2^m.

Theorem 1. *[5]*
1. The group G_k does not have normal subgroups of odd index $(\neq 1)$.
2. The group G_k has a normal subgroups of arbitrary even index.

2. New normal subgroups of finite index.

2.1. **The case of index eight.**

Definition 2. *A group G is called a **dihedral** group of degree 4 (i.e., D_4) if G is generated by two elements a and b satisfying the relations*

$$o(a) = 4, \ o(b) = 2, \ ba = a^3 b.$$

Definition 3. *A group G is called a **quaternion** group (i.e., Q_8) if G is generated by two elements a, b satisfying the relation*

$$o(a) = 4, a^2 = b^2, ba = a^3 b.$$

Remark 1. [2] *D_4 is not isomorph to Q_8.*

Definition 4. *A commutative group G is called a **Klein 8-group** (i.e., K_8) if G is generated by three elements a, b and c satisfying the relations $o(a) = o(b) = o(c) = 2$.*

Proposition 3. [2] *There exist (up to isomorphism) only two noncommutative nonisomorphic groups of order 8*

Proposition 4. *Let φ is a homomorphism of the group G_k onto a group G of order 8. Then $\varphi(G_k)$ is isomorph to either D_4 or K_8.*

Proof. `Case 1` Let $\varphi(G_k)$ is isomorph to any noncommutative group of order 8. By Proposition 1 $\varphi(G_k)$ is isomorph to either D_4 or Q_8. Let $\varphi(G_k) \simeq Q_8$ and e_q is an identity element of the group Q_8. Then $e_q = \varphi(e) = \varphi(a_i^2) = (\varphi(a_i))^2$ where $a_i \in G_k$, $i \in N_k$. Hence for the order of $\varphi(a_i)$ we have $o(\varphi(a_i)) \in \{1, 2\}$. It is easy to check there are only two elements of the group Q_8 which order of element less than two. This is contradict.

`Case 2` Let $\varphi(G_k, *)$ is isomorph to any commutative group $(G, *_1)$ of order 8. Then there exist distinct elements $a, b \in G$ such that $o(a) = o(b) = 2$. Let $H = \{e, a, b, ab\}$. It's easy to check that $(H, *_1)$ is a normal subgroup of the group $(G, *_1)$. For $c \notin H$ we have $H \neq cH$ ($cH = c *_1 H$). Hence $\varphi(G_k, *)$ is isomorph to only one commutative group $(cH \cup H, *_1)$. Clearly $(cH \cup H, *_1) \simeq K_8$. $\square$

The group G has a finitely generators of the order two and r is a minimal number of such generators of the group G and without loss of generality we can take these generators are $b_1, b_2, ... b_r$. Let e_1 is an identity element of the group G. We define homomorphism from G_k onto G. Let $\Xi_n = \{A_1, \ A_2, ..., A_n\}$ be a partition of $N_k \backslash A_0$, $0 \leq |A_0| \leq k + 1 - n$. Then we consider homomorphism $u_n : \{a_1, a_2, ..., a_{k+1}\} \to \{e_1, b_1 ..., b_m\}$ as

$$u_n(x) = \begin{cases} e_1, & \text{if } x = a_i, i \in A_0 \\ b_j, & \text{if } x = a_i, i \in A_j, j = \overline{1, n}. \end{cases} \tag{2.1}$$

For $b \in G$ we denote $R_b[b_1, b_2, ..., b_m]$ is a representation of the word b by generators $b_1, b_2, ..., b_r$, $r \leq m$. Define the homomorphism $\gamma_n : G \to G$ by the formula

$$\gamma_n(x) = \begin{cases} e_1, & \text{if } x = e_1 \\ b_i, & \text{if } x = b_i, i = \overline{1, r} \\ R_{b_i}[b_1, ..., b_r], & \text{if } x = b_i, i \neq \overline{1, r} \end{cases} \tag{2.2}$$

Put

$$H_{\Xi_n}^{(p)}(G) = \{x \in G_k | \ l(\gamma_n(u_n(x))) : 2p\}, \ 2 \leq n \leq k - 1. \tag{2.3}$$

Let $\gamma_n(u_n(x))) = \tilde{x}$. We introduce the following equivalence relation on the set G_k : $x \sim y$ if $\tilde{x} = \tilde{y}$. It's easy to check this relation is reflexive, symmetric and transitive.

Proposition 5. *Let $\Xi_n = \{A_1, A_2, ..., A_n\}$ be a partition of $N_k \backslash A_0$, $0 \le |A_0| \le k + 1 - n$. Then $H_{\Xi_p}^{(p)}(G)$ is a normal subgroup of index $2p$ of the group G_k.*

Proof. For $x = a_{i_1} a_{i_2} ... a_{i_n} \in G_k$ it's sufficient to show that $x^{-1} H_{\Xi_n}^{(p)}(G)\ x \subseteq H_{\Xi_n}^{(p)}(G)$. Suppose that there exist $y \in G_k$ such that $l(\tilde{y}) \ge 2p$. Let $\tilde{y} = b_{i_1} b_{i_2} ... b_{i_q}$, $q \ge 2p$ and $S = \{b_{i_1}, b_{i_1} b_{i_2}, ..., b_{i_1} b_{i_2} ... b_{i_q}\}$. Since $S \subseteq G$ there exist $x_1, x_2 \in S$ such that $x_1 = x_2$. But this is contradict to $\tilde{y}$ is a non-contractible. Thus we have proved that $l(\tilde{y}) < 2p$. Hence for any $x \in H_{\Xi_n}^{(p)}(G)$ we have $\tilde{x} = e_1$. Now we take any element z from the set $x^{-1} H_{\Xi_n}^{(p)}(G)\ x$. Then $z = x^{-1} h\ x$ for some $h \in H_{\Xi_n}^{(p)}(G)$. We have $\tilde{z} = \gamma_n(v_n(z)) = \gamma_n\left(v_n(x^{-1} h\ x)\right) = $ $= \gamma_n\left(v_n(x^{-1}) v_n(h) v_n(x)\right) = \gamma_n\left(v_n(x^{-1})\right) \gamma_n\left(v_n(h)\right) \gamma_n\left(v_n(x)\right)$. From $\gamma_n\left(v_n(h)\right) = e_1$ we get $\tilde{z} = e_1$ i.e., $z \in H_{\Xi_n}^{(p)}(G)$. This completes the proof. $\square$

For $A_1, A_2, A_3 \subset N_k$ and $\cap_{i=1}^3 H_{A_i}$ is non-contractible we denote following set

$$\Re = \{\cap_{i=1}^3 H_{A_i} |\ A_1, A_2, A_3 \subset N_k\}$$

Theorem 2. *For the group G_k following statement is hold*

$$\{H|\ H \text{ is a normal subgroup of } G_k \text{ with } |G_k : H| = 8\} =$$
$$= \{H_{C_0 C_1 C_2}^{(4)}(D_4) |\ C_1, C_2 \text{ is a partition of } N_k \setminus C_0\} \cup \Re.$$

Proof. Let ϕ be a homomorphism with $|G_k : Ker\phi| = 8$. Then by Proposition 2 we have $\phi(G_k) \simeq K_8$ and $\phi(G_k) \simeq D_4$.

Let $\phi : G_k \to K_8$ be an epimorphism. For any nonempty sets $A_1, A_2, A_3 \subset N_k$ we give one to one correspondence between $\{Ker\phi|\ \phi(G_k) \simeq K_8\}$ and $\Re$. Let $a_i \in G_k, i \in N_k$. We define following homomorphism corresponding to the set A_1, A_2, A_3.

$$\phi_{A_1 A_2 A_3}(a_i) = \begin{cases} a, & \text{if } i \in A_1 \setminus (A_2 \cup A_3) \\ b, & \text{if } i \in A_2 \setminus (A_1 \cup A_3) \\ c, & \text{if } i \in A_3 \setminus (A_1 \cup A_2) \\ ab, & \text{if } i \in (A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3) \\ ac, & \text{if } i \in (A_1 \cap A_3) \setminus (A_1 \cap A_2 \cap A_3) \\ bc, & \text{if } i \in (A_2 \cap A_3) \setminus (A_1 \cap A_2 \cap A_3) \\ abc, & \text{if } i \in A_1 \cap A_2 \cap A_3 \\ e, & \text{if } i \in N_k \setminus (A_1 \cup A_2 \cup A_3). \end{cases}$$

If $i \in \emptyset$ then we'll accept that there is not any index $i \in N_k$ which that condition is not satisfied. It is easy to check $Ker\phi_{A_1 A_2 A_3} = H_{A_1} \cap H_{A_2} \cap H_{A_3}$. Hence $\{Ker\phi|\ \phi(G_k) \simeq K_8\} = \Re$.

Now we'll consider the case $\phi(G_k) \simeq D_4$. Let $\phi : G_k \to D_4$ be epimorphisms. Put

$$C_0 = \{i|\ \phi(a_i) = e\}, \quad C_1 = \{i|\ \phi(a_i) = b\}, \quad C_2 = \{i|\ \phi(a_i) = ab\}$$

One can construct following homomorphism (corresponding to C_0, C_1, C_2)

$$\phi_{C_0 C_1 C_2}(x) = \begin{cases} e, & \text{if } \tilde{x} = e \\ a, & \text{if } \tilde{x} = b_2 b_1 \\ a^2, & \text{if } \tilde{x} = b_2 b_1 b_2 b_1 \\ a^3, & \text{if } \tilde{x} = b_2 b_1 b_2 b_1 b_2 b_1 \\ b, & \text{if } \tilde{x} = b_1 \\ ab, & \text{if } \tilde{x} = b_2 \\ a^2 b, & \text{if } \tilde{x} = b_2 b_1 b_2 \\ a^3 b, & \text{if } \tilde{x} = b_2 b_1 b_2 b_1 b_2. \end{cases}$$

Straight away we conclude $Ker(\phi_{C_0 C_1 C_2}) = H^{(4)}_{C_0 C_1 C_2}(D_4)$. We have constructed all homomorphisms ϕ on the group G_k which $|G_k : Ker\phi| = 8$. Thus by Proposition 1 one get

$$\{H| \ |G_k : H| = 8\} \subseteq \{H^{(4)}_{C_0 C_1 C_2}(D_4)| \ C_1, C_2 \ is \ a \ partition \ of \ N_k \setminus C_0\} \cup \Re.$$

By Proposition 2 and Proposition 5 we can see easily

$$\Re \cup \{H^{(4)}_{C_0 C_1 C_2}(D_4)| \ C_1, C_2 \ is \ a \ partition \ of \ N_k \setminus C_0\} \subseteq \{H| \ |G_k : H| = 8\}.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Corollary 1. *The number of all normal subgroups of index 8 for the group G_k is equal to* $8^{k+1} - 6 \cdot 4^{k+1} + 3^{k+1} + 9 \cdot 2^{k+1} - 5$.

Proof. Number of elements of the set $H_A \subset G_k, \emptyset \neq A \subset N_k$ is $2^{k+1} - 1$. Then $|\Re| = (2^{k+1})(2^{k+1} - 2)(2^{k+2} - 3)$. Let $C_0 \subset N_k$ is a fixed set and $|C_0| = p$. If C_1, C_2 is a partition of $N_k \setminus C_0$ then there are $2^{k-p+1} - 2$ ways to choose the sets C_1 and C_2. Hence the cardinality of $\{H^{(4)}_{C_0 C_1 C_2}(D_4)| \ C_1, C_2 \ is \ a \ partition \ of \ N_k \setminus C_0\}$ is equal to

$$(2^{k+1} - 2)C^0_{k+1} + (2^k - 2)C^1_{k+1} + ... + 2C^{k-1}_{k+1} = 3^{k+1} - 2^{k+2} + 1.$$

Since $\Re$ and $\{H^{(4)}_{C_0 C_1 C_2}(D_4)\}| \ C_1, C_2 \subset N_k$ are disjoint sets, cardinality of the union of these sets is $8^{k+1} - 6 \cdot 4^{k+1} + 3^{k+1} + 9 \cdot 2^{k+1} - 5$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

2.2. Case of index ten. Let the group R_{10} is generated by the permutations

$$\pi_1 = (1,2)(3,4)(5,6), \ \pi_2 = (2,3)(4,5)$$

Proposition 6. *Let φ is a homomorphism of the group G_k onto a group G of order 10. Then $\varphi(G_k)$ is isomorph to R_{10}.*

Proof. Let $(G, *)$ be a group and $|G| = 10$. Suppose there exist an epimorphism from G_k onto G. It is easy to check that there are at least two elements $a, b \in G_k$ such that $o(a) = o(b) = 2$. If $a * b = b * a$ then $(H, *)$ is a subgroup of the group $(G, *)$, where $H = \{e, a, b, a * b\}$. Then by Lagrange's theorem $|G|$ is divide by $|H|$ but 10 is not divide by 4. Hence $a * b \neq b * a$. We have $\{e, a, b, a * b, b * a\} \subset G$ If G is generated by three elements then there exist an element c such that $c \notin \{e, a, b, a * b, b * a\}$. Then the set $\{e, a, b, a * b, b * a, c, c * a, c * b, c * a * b, c * b * a\}$ must be equal to G. Since G is a group we get $a * b * a = b$ but from $o(a) = 2$ the last equality is equivalent to $a * b = b * a$. This is a contradict. Hence by Lagrange is theorem it's easy to see

$$G = \{e, a, b, a * b, b * a, a * b * a, b * a * b, a * b * a * b, b * a * b * a, a * b * a * b * a\},$$

where $o(a * b) = 5$. Namely $G \simeq R_{10}$. This completes the proof. $\square$

Theorem 3. *For the group G_k following statement is hold*

$$\{H| \mid H \text{ is a normal subgroup of } G_k \text{ with } |G_k : H| = 10\} =$$

$$= \{H^{(5)}_{B_0 B_1 B_2}(R_{10})| \ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\}.$$

Proof. Let ϕ be a homomorphism with $|G_k : Ker\phi| = 10$. By Proposition 6 $\phi(G_k) \simeq R_{10}$ and by Proposition 5 we can see easily

$$\{H^{(5)}_{B_0 B_1 B_2}(R_{10})| \ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\} \subset \{H| \ |G_k : H| = 10\}.$$

Let $\varphi : G_k \to R_{10}$ be epimorphisms. Denote

$$B_0 = \{i| \ \varphi(a_i) = e\}, \quad B_1 = \{i| \ \varphi(a_i) = a, \ B_2 = \{i| \ \varphi(a_i) = b\}.$$

Then we can show this homomorphism (corresponding to $B_1, \ B_2, \ B_3$), i.e.,

$$\phi_{B_0 B_1 B_2}(x) = \begin{cases} e, & \text{if } \tilde{x} = e \\ a, & \text{if } \tilde{x} = b_1 \\ b, & \text{if } \tilde{x} = b_2 \\ a * b, & \text{if } \tilde{x} = b_1 b_2 \\ b * a, & \text{if } \tilde{x} = b_2 b_1 \\ a * b * a, & \text{if } \tilde{x} = b_1 b_2 b_1 \\ b * a * b, & \text{if } \tilde{x} = b_2 b_1 b_2 \\ a * b * a * b, & \text{if } \tilde{x} = b_1 b_2 b_1 b_2 \\ b * a * b * a, & \text{if } \tilde{x} = b_2 b_1 b_2 b_1 \\ a * b * a * b * a, & \text{if } \tilde{x} = b_1 b_2 b_1 b_2 b_1. \end{cases}$$

We have constructed all homomorphisms ϕ on the group G_k which $|G_k : Ker\phi| = 10$. Hence

$$\{Ker\phi| \ |G_k : Ker\phi| = 10\} \subset \{H_{B_0B_1B_2}^{(5)}(R_{10})| \ B_1, B_2 \ is \ a \ partition \ of \ the \ set \ N_k \setminus B_0\}.$$

By Proposition 1

$$\{H| \ |G_k : H| = 10\} = \{H_{B_0B_1B_2}^{(5)}(R_{10})| \ B_1, B_2 \ is \ a \ partition \ of \ the \ set \ N_k \setminus B_0\}.$$

The theorem is proved. $\qquad\square$

Corollary 2. *The number of all normal subgroups of index* 10 *for the group* G_k *is equal to* $3^{k+1} - 2^{k+2} + 1$.

Proof. To prove this Corollary is similar to proof of Corollary 1. $\qquad\square$

Acknowledgements

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